LAW OF SINES AND LAW OF COSINES

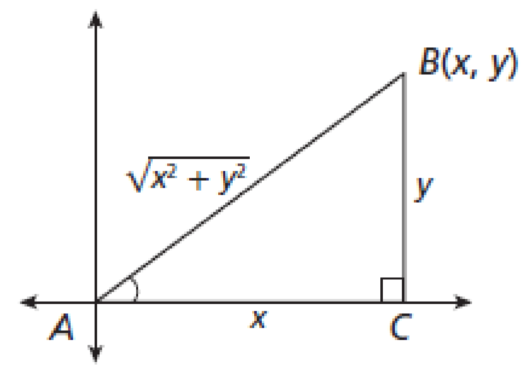
8-5

TO USE THE LAW OF SINES AND/OR THE LAW OF COSINES TO FIND ANGLE MEASURES AND SIDE LENGTHS IN NON-RIGHT TRIANGLES

OBJECTIVE

NOTES

A First, extend the trigonometric ratios to right triangles on a coordinate plane.



Place right triangle $\triangle ABC$ with acute $\angle A$ on a coordinate plane as shown. Let the coordinates of B be $B(x, y)$. Then the lengths of the legs are x and y.

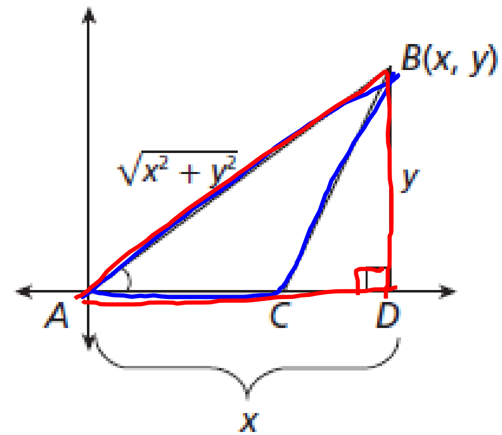
The length of the hypotenuse is $\sqrt{x^2 + y^2}$ by Pythagorean Theorem.

Now the trigonometric ratios for $\angle A$ can be expressed in terms of x and y.

$\sin A = \frac{y}{\sqrt{x^2+y^2}}$ $\cos A = \frac{x}{\sqrt{x^2+y^2}}$ $\tan A = \frac{y}{x}$

NOTES

B Next, extend the trigonometric ratios to acute angles in non-right triangles.



Place non-right triangle $\triangle ABC$ with acute $\angle A$ on a coordinate plane as shown. Let the coordinates of B be $B(x, y)$.

Draw a perpendicular from B to the x-axis, and label the point of intersection D. Then the lengths of the legs in right triangle $\triangle ABD$ are x and y.

The length of the hypotenuse is $\sqrt{x^2 + y^2}$ by Pythagorean Theorem.

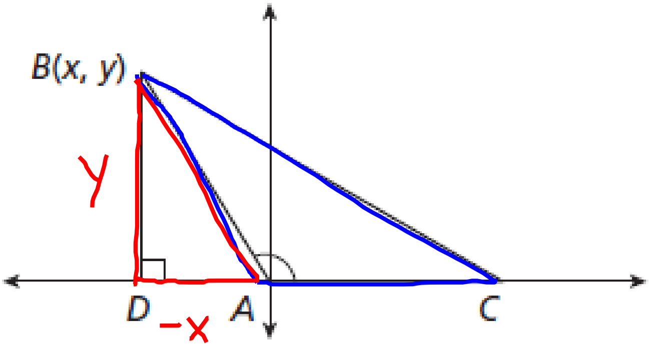
Define the trigonometric ratios for $\angle A$ in terms of x and y by using the side lengths of $\triangle ABD$.

$\sin A = \frac{y}{\sqrt{x^2 + y^2}}$ $\cos A = \frac{x}{\sqrt{x^2 + y^2}}$ $\tan A = \frac{y}{x}$

NOTES

C Finally, extend the trigonometric ratios to obtuse angles in non-right triangles.

Place non-right triangle $\triangle ABC$ with obtuse $\angle A$ on a coordinate plane as shown. Let the coordinates of B be $B(x, y)$.



Draw a perpendicular from B to the x-axis, and label the point of intersection D. Then let the "lengths" of the legs in right triangle $\triangle ABD$ be x and y, where it is understood that $x < 0$.

The length of the hypotenuse, $\overline{AB}$, is $\sqrt{x^2+y^2}$.

Define the trigonometric ratios for $\angle A$ in terms of x and y by using the sides of $\triangle ABD$.

$\sin A = \dfrac{y}{\sqrt{x^2+y^2}}$ $\qquad \cos A = \dfrac{-x}{\sqrt{x^2+y^2}}$ $\qquad \tan A = \dfrac{y}{-x}$

NOTES

D You can use a calculator to find trigonometric ratios for obtuse angles. Use a calculator to complete the table below. Round to the nearest hundredth.

83° .9925 .1219 8.1443

Angle	Sine	Cosine	Tangent
97°	.9925	−.1219	−8.1443
122°	.8480	−.5299	−1.6003
165°	.2588	−.9659	−.2679

REFLECT

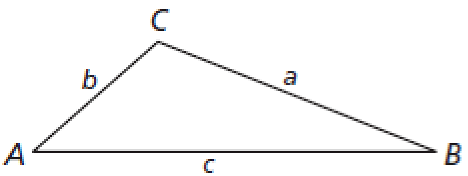
1a. Look for patterns in your table. Make a conjecture about the trigonometric ratios of obtuse angles.

NOTES

You can use sines and cosines to solve problems that involve non-right triangles. One example is the Law of Sines, which is a relationship that holds for any triangle.

Law of Sines

For $\triangle ABC$, $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$.



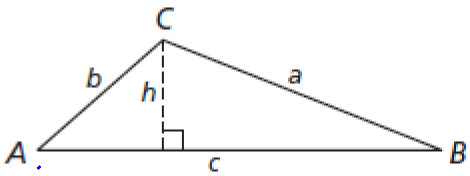
You can use the Law of Sines to solve a triangle when you are given the following information.

- Two angle measures and any side length (AAS or ASA information).
- Two side lengths and the measure of a non-included angle (SSA information).

NOTES

G-SRT.4.10(+)

2 PROOF **The Law of Sines**



Complete the proof.

Given: $\triangle ABC$

Prove: $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$

A Draw an altitude from C to side $\overline{AB}$. Let h be the length of the altitude.

Then $\sin A = \frac{h}{b}$ and $\sin B = \frac{h}{a}$.

Solve the two equations for h.

$h =$ $b \sin A$ and $h =$ $a \sin B$

B Write a new equation by setting the right sides of the above equations equal to each other.

$\frac{\cancel{b} \sin A}{a\cancel{b}} = \frac{\cancel{a} \sin B}{\cancel{a}b}$ Substitute.

$\frac{\sin A}{a} = \frac{\sin B}{b}$ Divide both sides by ab.

Similar reasoning shows that $\frac{\sin A}{a} = \frac{\sin C}{c}$ and $\frac{\sin B}{b} = \frac{\sin C}{c}$.

EXAMPLE

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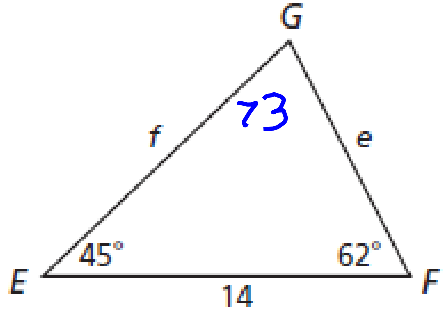
3 EXAMPLE **Using the Law of Sines**

Solve the triangle. Round to the nearest tenth.



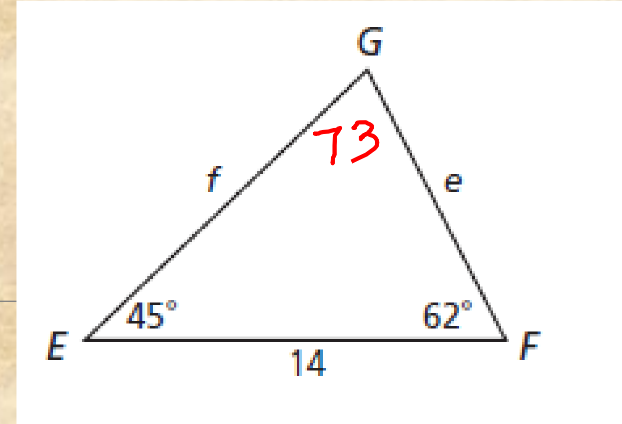
A Find the unknown angle measure.

$m\angle E + m\angle F + m\angle G = 180°$ Triangle Sum Theorem

$45° + 62° + m\angle G = 180°$ Substitute.

$m\angle G =$ 73 Solve for $m\angle G$.

EXAMPLE



B Use the Law of Sines to find the unknown side length *e*.

$$\frac{\sin E}{e} = \frac{\sin G}{g}$$ Law of Sines

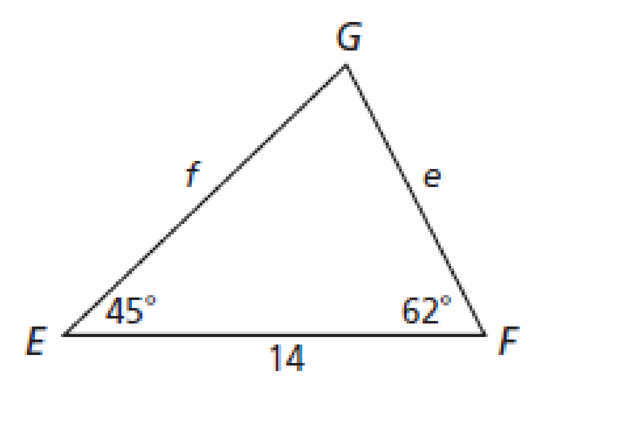
$$\frac{\sin 45^\circ}{e} = \frac{\sin 73}{14}$$ Substitute.

$$14 \sin 45^\circ = e \cdot \sin(73)$$ Multiply both sides by the product of the denominators.

$$\frac{14 \cdot \sin 45^\circ}{\sin 73} = e$$ Solve for e.

$$e \approx 10.4$$ Use a calculator to evaluate. Round.

EXAMPLE



C Use the Law of Sines to find the unknown side length *f*.

$$\frac{\sin F}{f} = \frac{\sin G}{g}$$ Law of Sines

$$\frac{\sin 62°}{f} = \frac{\sin 73}{14}$$ Substitute.

$$14 \sin 62° = f \cdot \sin(23)$$ Multiply both sides by the product of the denominators.

$$\frac{14 \cdot \sin 62°}{\sin 73} = f$$ Solve for *f*.

$$f \approx 12.9$$ Use a calculator to evaluate. Round.

NOTES

When you are given SSS or SAS information about a triangle, you cannot use the Law of Sines to solve the triangle. However, this information determines a unique triangle, so there should be some way to find the unknown side lengths and angle measures. The Law of Cosines is useful in this case.

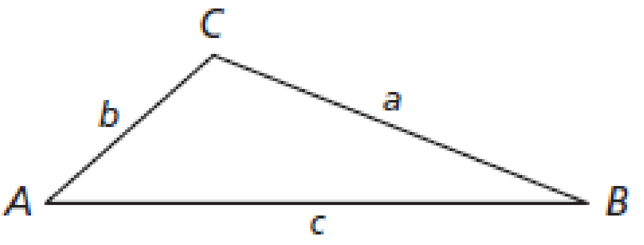
Law of Cosines

For $\triangle ABC$,

$a^2 = b^2 + c^2 - 2bc \cos A,$

$b^2 = a^2 + c^2 - 2ac \cos B,$

$c^2 = a^2 + b^2 - 2ab \cos C.$



NOTES

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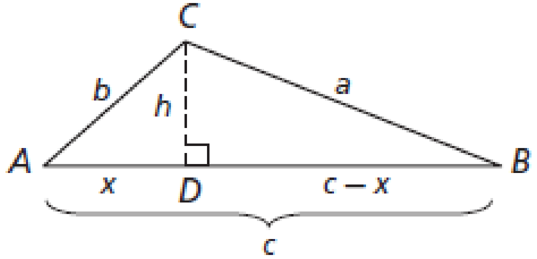
4 PROOF The Law of Cosines

Complete the proof.

Given: $\triangle ABC$

Prove: $a^2 = b^2 + c^2 - 2bc \cos A$



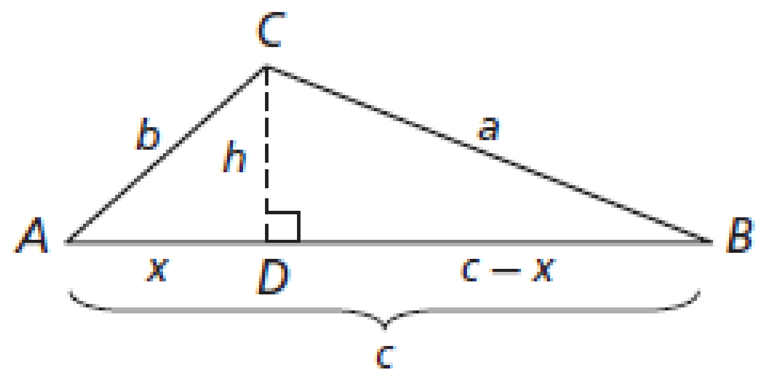
A Draw an altitude $\overline{CD}$ to side $\overline{AB}$. Let h be the length of the altitude.

Let x be the length of $\overline{AD}$. Then $c - x$ is the length of $\overline{DB}$.

In $\triangle ADC$, $\cos A = \frac{x}{b}$ and so $x =$ $b \cos A$.

Also, by the Pythagorean Theorem, $x^2 + h^2 =$ b^2.

NOTES



B Now consider $\triangle CDB$.

$(c-x)(c-x)$

$a^2 = (c - x)^2 + h^2$ Pythagorean Theorem

$c^2 - cx - cx + x^2$

$c^2 - 2cx + x^2$

$a^2 = c^2 - 2cx + (x^2 + h^2)$ Expand $(c - x)^2$.

$a^2 = c^2 - 2cx +$ <u>b^2</u> Substitute <u>b^2</u> for $x^2 + h^2$.

$a^2 = b^2 + c^2 - 2cx$ Rearrange terms.

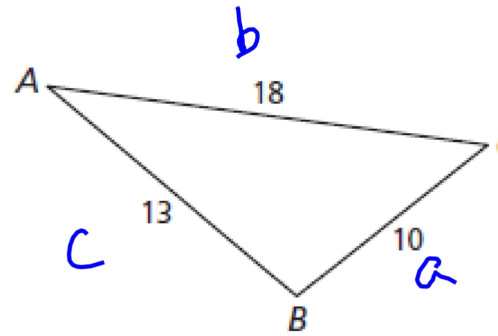
$a^2 = b^2 + c^2 - 2c($<u>$b\cos A$</u>$)$ Substitute <u>$b\cos A$</u> for x.

Similar reasoning shows that $b^2 = a^2 + c^2 - 2ac\cos B$ and $c^2 = a^2 + b^2 - 2ab\cos C$.

EXAMPLE

G-SRT.4.11(+)

5 EXAMPLE **Using the Law of Cosines**



Solve the triangle. Round to the nearest tenth.

A Find the measure of the obtuse angle first.

$b^2 = a^2 + c^2 - 2ac\cos B$ Law of Cosines

$18^2 = 10^2 + 13^2 - 2(10)(13)\cos B$ Substitute.

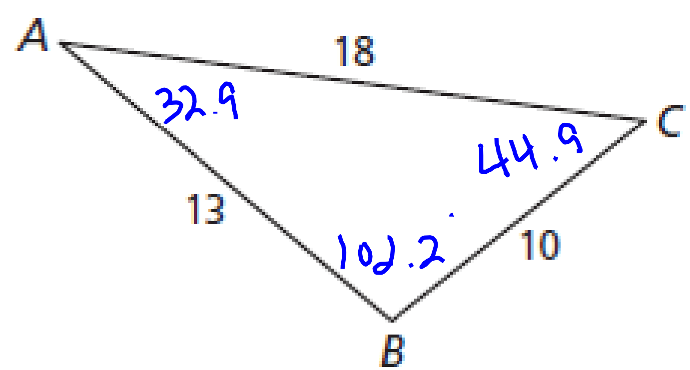
$18^2 - 10^2 - 13^2 = -2(10)(13)\cos B$

$\cos B = \frac{18^2 - 10^2 - 13^2}{-2(10)(13)} = \frac{55}{-260}$ Solve for cos *B*.

$\cos B \approx -.2115$ Simplify. Round to four decimal places.

$m\angle B \approx \cos^{-1}(-.2115) \approx 102.2^\circ$ Solve for $m\angle B$.

EXAMPLE



B Use the Law of Sines to find m∠C.

$\frac{\sin C}{c} = \frac{\sin B}{b}$ Law of Sines

$\frac{\sin C}{13} = \frac{\sin 102.2125}{18}$ Substitute.

$\sin C = \frac{13 \sin 102.2125}{18} \approx .7059$ Multiply both sides by 13, and then simplify.

$m\angle C \approx \sin^{-1}(.7059) \approx 44.9$ Solve for m∠C.

C Use the Triangle Sum Theorem to find the remaining angle measure.

$m\angle A \approx 180° - 102.2 - 44.9 = 32.9$

HOMEWORK

Thursday: Page 573: 2 – 16 even

Friday: Pages 573 – 574: 26 - 38 even